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ON

# THE APPLICATION

OF A

# NEW ANALYTIC METHOD

TO

# THE THEORY

OF

# CURVES AND CURVED SURFACES.

BY THE

REV. JAMES BOOTH, LL.D., M.R.I.A.,

PROFESSOR OF MATHEMATICS IN THE COLLEGIATE INSTITUTION, LIVERPOOL,

AND CHAPLAIN TO THE MOST NOBLE THE MARQUESS OF LANSDOWNE, ETC.

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TO THE REVEREND

FRANC SADLEIR, D.D.,

PROVOST OF TRINITY COLLEGE, DUBLIN,

THE FOLLOWING PAGES ARE DEDICATED

AS A TRIBUTE OF RESPECT,

BY THE

AUTHOR.

# ADVERTISEMENT.

I FEAR that brevity and compression have been but too much studied in the following essay, but the necessity of comprising the whole matter in a small compass, and the pressure of other avocations, will plead, I hope, a sufficient apology.

From the same cause I have been obliged to omit altogether subjects which might have been with propriety introduced, for example, the general theory of *shadows;* and have only touched upon others which would require perhaps further development.

Among other applications of the method, I trust that to the theory of *reciprocal polars* will be found simple and satisfactory.

My attention has been just directed by a friend to a letter from M. Chasles, dated December 10, 1829, published in the *Correspondence Mathématique* of M. Quetelet, tom. vi. p. 81, in which the writer asserts his claim to the invention of a system of coordinates, noticed by M. Plucker in one of the livraisons of Crelle's Journal, to which work I have never had an opportunity of referring. After some preliminary observations, he states his system as follows:—" Pour cela, par trois points fixes A, B, C, je mène trois axes parallèles entre eux, un plan quelconque rencontre ces axes en trois points dont les distances aux points A, B, C, respectivement, sont les coordonnées $x, y, z$, du plan," &c.; and then goes on to apply his system to a few examples, using the principles and notation of the differential calculus. To any one consulting the letter from which the above extract is taken, it will be apparent that the method there proposed, however excellent and ingenious it may be, bears not the least resemblance to the one developed in the following pages.

Some valuable improvements in the notation I have adopted, have been suggested by the Reverend Charles Graves, F. T. C. D., of which I have thankfully availed myself.

J. B.

TRINITY COLLEGE, DUBLIN,
*March* 25*th*, 1840.

ON

# TANGENTIAL COORDINATES.

## CHAPTER I.

It must have often appeared an anomalous fact in the application of algebraic analysis to geometrical investigations, that while the locus of a point could be found from the simplest and most elementary considerations, the envelope of a right line or plane could be determined only by the aid of principles, artificial and obscure, derived from a higher department of analysis.

But this is not the only or the greatest objection to the method at present universally followed—it is in most cases operose, and in some impracticable, to reduce the equation $V=0$ to the form $\frac{dV}{d\alpha}=0$ and then eliminate the auxiliary variable $\alpha$, between those equations, a difficulty which becomes far more formidable in problems of three dimensions, where we are obliged to eliminate the auxiliary variables $\alpha$ and $\beta$ between the three equations

$$V=0, \quad \frac{dV}{d\alpha}=0, \quad \frac{dV}{d\beta}=0.$$

As it follows *à priori* from the principle of *duality*,* that for every locus of a point there exists a corresponding envelope of a right line or plane, it would seem that the comparative paucity of theorems

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* See various memoirs on this subject by MM. Gergonne, Poncelet, and others, dispersed through the volumes of the *Annales de Mathématiques*.

of the latter species generally known, can be owing to nothing but the want of a simple and direct mode of investigation.

From these considerations I have been led to the discovery of a method simple in principle, and easy of application, analogous to, but different from, that of rectilinear or *projective* coordinates,—as for distinction they may be called,—in which the reciprocals of the distances of the origin from the points where the axes of coordinates are met by a right line, or plane, touching a curve or curved surface, are denoted by the letters $\xi$, $\upsilon$, $\zeta$; an equation established between them, may be called the *tangential equation* of the curve or curved surface.

By the help of this equation we may elude the necessity of differentiating the equation $V = 0$, and discover the envelopes of right lines and planes with the same facility as the locus of a point by projective coordinates.

But it is not alone in inquiries of this nature that the method is chiefly valuable; there is a large class of theorems relating to curves touching given right lines, and surfaces in contact with given planes, which may be treated by the method proposed with the greatest facility, whose solution by projective coordinates would lead to exceedingly complicated and unmanageable expressions.

## I.

### ON THE TANGENTIAL EQUATION OF A POINT IN A PLANE.

Through any point in the plane let two rectangular axes OX, OY, be drawn, let $\alpha$, $\beta$, denote the projective coordinates of the point on those axes, $\xi$ and $\upsilon$ the reciprocals of the intercepts of the axes OX, OY, from the origin made by any line passing through the point, the position of the point is determined by the equation

$$\alpha\,\xi + \beta\,\upsilon = 1. \tag{1}$$

The tangential equations of a right line are

$$\xi = \text{constant}, \qquad \upsilon = \text{constant}. \tag{2}$$

## II.

### ON THE TRANSFORMATION OF COORDINATES.

Let $\frac{1}{\xi}$ and $\frac{1}{\upsilon}$ denote the intercepts of two rectangular axes OX, OY, by a given right line, $\frac{1}{\xi'}$ and $\frac{1}{\upsilon'}$ the intercepts by the same right line

of two other rectangular axes OX′, OY′, making an angle $\theta$ with the former, then

$$\xi = \xi' \cos\theta + \upsilon' \sin\theta ; \qquad \upsilon = \upsilon' \cos\theta - \xi' \sin\theta . \tag{3}$$

The axes remaining parallel, let the origin be translated to a point whose projective coordinates are $p$ and $q$, let $\frac{1}{\xi'}$ and $\frac{1}{\upsilon'}$ be the intercepts of the new axes; then

$$\xi = \frac{\xi'}{1 + p\,\xi' + q\,\upsilon'}, \qquad \upsilon = \frac{\upsilon'}{1 + p\,\xi' + q\,\upsilon'}; \tag{4}$$

hence $\frac{\upsilon}{\xi} = \frac{\upsilon'}{\xi'}$.

III.

ON THE TANGENTIAL EQUATIONS OF THE CENTRAL CONIC SECTIONS.

The projective equation of a conic section referred to its centre and axes being

$$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1 ; \tag{a}$$

the equation of a tangent to it is $\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1$. when $y = 0$, $x = \frac{1}{\xi}$, when $x = 0$, $y = \frac{1}{\upsilon}$, $\therefore x' = a^2\,\xi$, $y' = b^2\,\upsilon$, substituting in (a) the tangential equation of the section becomes

$$a^2\,\xi^2 + b^2\,\upsilon^2 = 1 . \tag{5}$$

Let the axes of coordinates now be conceived to revolve through an angle $\theta$ round O, and then translated to a point of which the projective coordinates are $p$ and $q$, by the aid of formulæ (3) and (4) equation (5) is transformed into

$$\left.\begin{aligned} &[a^2 \cos^2\theta + b^2 \sin^2\theta - p^2]\,\xi^2 + [b^2 \cos^2\theta + a^2 \sin^2\theta - q^2]\,\upsilon^2 \\ &\quad + 2\,[(a^2 - b^2) \sin\theta \cos\theta - p\,q]\,\xi\,\upsilon - 2\,p\,\xi - 2\,q\,\upsilon = 1 ; \end{aligned}\right\} \tag{6}$$

hence the tangential equation of a conic section being given in the general form

$$\alpha\,\xi^2 + \alpha'\,\upsilon^2 + 2\beta\,\xi\,\upsilon + 2\gamma\,\xi + 2\gamma'\,\upsilon = 1. \tag{7}$$

Comparing its coefficients with those of (6) we shall have five equations, to determine the projective coordinates ef the centre, the magnitude and inclination of the axes of the section. In the first place $\gamma = -p$, $\gamma' = -q$; hence one half the coefficients of the linear

terms in the tangential equation are the projective coordinates of the centre.

Comparing the three remaining coefficients, and introducing the values of $p$ and $q$ we have

$$a^2 \cos^2 \theta + b^2 \sin^2 \theta = a + \gamma^2 ; \quad b^2 \cos^2 \theta + a^2 \sin^2 \theta = a'^2 + \gamma'^2 ;$$

and $(a^2 - b^2) \sin \theta \cos \theta = \beta + \gamma\gamma'$ ;

hence
$$\tan 2\theta = \frac{2(\beta + \gamma\gamma')}{(a + \gamma^2) - (a' + \gamma'^2)} ; \tag{8}$$

$$a^2 = \frac{a + a' + \gamma^2 + \gamma'^2 \pm \sqrt{\{(a + \gamma^2) - (a' + \gamma'^2)\}^2 + 4(\beta + \gamma\gamma')^2}}{2} . \tag{9}$$

The curve is an ellipse or hyperbola according as

$$(\beta + \gamma\gamma')^2 < \text{or} > \text{than} \, (a + \gamma^2)(a' + \gamma'^2).$$

Let $\tau =$ tangent of the angle which one of the asymptots of an hyperbola makes with the axis of $x$, then

$$\tau = \frac{-(\beta + \gamma\gamma') \pm \sqrt{(\beta + \gamma\gamma')^2 - (a + \gamma^2)(a' + \gamma'^2)}}{a} . \tag{10}$$

When the two conditions

$$a + \gamma^2 = a' + \gamma'^2, \text{ and } (\beta + \gamma\gamma') = 0,$$

are satisfied, the curve is a circle, and the origin is at a focus when $a = a'$ and $\beta = 0$.

Now the projective equation of a conic section being

$$\text{A}\, x^2 + \text{A}' y^2 + 2\text{B}\, xy + 2\text{C}\, x + 2\text{C}' y = 1,$$

it may be shown that the origin is at a focus when $\text{A} + \text{C}^2 = \text{A}' + \text{C}'^2$ and $\text{B} + \text{CC}' = 0$, and the curve is a circle when $\text{A} = \text{A}'$ and $\text{B} = 0$, which conditions are reciprocally analogous to those just mentioned.

The origin of coordinates is on the curve when $aa' - \beta^2 = 0$.

When $a' = 0$ the curve touches the axis of $x$, when $a = 0$ it touches the axis of $y$.

Let the general tangential equation of a conic senction

$$a\,\xi^2 + a'\,v^2 + 2\beta\,\xi\,v + 2\gamma\,\xi + 2\gamma' v = 1,$$

be solved for $v$,

$$v = -\frac{(\beta\xi + \gamma')}{a'} \pm \frac{\sqrt{\text{M}}}{a'}.$$

Now let
$$\frac{1}{om} = -\frac{(\beta\,\xi + \gamma')}{a'} + \frac{\sqrt{M}}{a'},$$
$$\frac{1}{op} = -\frac{(\beta\,\xi + \gamma')}{a'},$$
$$\frac{1}{on} = -\frac{(\beta\,\xi + \gamma')}{a'} - \frac{\sqrt{M}}{a'};$$
then $\frac{1}{om}$, $\frac{1}{op}$, $\frac{1}{on}$ are in arithmetical progression $\because$ $om$. $op$. $on$ are in harmonic; from these principles it easily follows that
$$a'\,v + \beta\,\xi + \gamma' = 0, \tag{11}$$
is the tangential equation of the pole of the axis of $x$. In the same manner
$$a\,\xi + \beta\,v + \gamma = 0, \tag{12}$$
is the equation of the pole of the axis of $y$; hence the simultaneous equations $a'\,v + \beta\,\xi + \gamma' = 0$, and $a\,\xi + \beta\,v + \gamma = 0$, are the tangential equations of the polar* of the origin.

The analogy between those equations, and the equations which determine the centre in projective coordinates is manifest; we shall now apply this theory to some examples.

*The product of perpendiculars let fall from two points on a right line is constant; the line envelopes a conic section.*

Let the line joining the points whose distance is $2c$, be taken as the axis of $x$, the origin being at the middle point, then $P = \frac{(1 - c\xi)}{\sqrt{\xi^2 + v^2}}$, $P' = \frac{(1 + c\xi)}{\sqrt{\xi^2 + v^2}}$; but $PP' = b^2$ suppose
$$\therefore\ (c^2 + b^2)\,\xi^2 + b^2\,v^2 = 1,$$
which is the tangential equation of a conic section.

*The vertex of a right angle moves along a given circle, one side passes through a fixed point, the other envelopes a conic section.*

Let the line joining the fixed point with the centre of the circle—the origin of coordinates—be taken as the axis of $x$, the distance of this point from the centre $= c$; then we shall have

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* A right line being given in the plane of a conic section, if from any point in it tangents are drawn to the curve, the cords of contact pass through a fixed point, this point and the given right line are termed *pole* and *polar* relative to the conic section.

$$x = \frac{c\,v^2 - \xi}{\xi^2 + v^2}, \qquad y = \frac{v\,(1 + c\,\xi)}{\xi^2 + v^2};$$

substituting those values of $x$ and $y$ in the equation of the circle $x^2 + y^2 = a^2$ we find

$$a^2\,\xi^2 + (a^2 - c^2)\,v^2 = 1.$$

*Conic sections are inscribed in the same quadrilateral, the polar of any point in their plane envelopes a conic section.*

The fixed point being assumed as origin of coordinates let the tangential equation of one of the sections be

$$a\,\xi^2 + a'v^2 + 2\beta\,\xi\,v + 2\gamma\,\xi + 2\gamma' v = 1. \qquad \text{(a)}$$

The equations of the polar of the origin are (11,.12)

$$a\,\xi + \beta\,v + \gamma = 0, \text{ (b), and } a'\,v + \beta\xi + \gamma' = 0, \text{ (b}'\text{)},$$

now there are four linear equations to determine the five unknown quantities $a$, $a'$, $\beta$, $\gamma$, $\gamma'$, we may then eliminate any three and connect the fourth and fifth by a linear equation, eliminating then $a$, $a'$, $\gamma$, $\gamma'$, three by three successively, we have

$$a = \mathrm{K}\,\beta + \mathrm{L}, \qquad a' = \mathrm{K}'\beta + \mathrm{L}',$$
$$\gamma = \mathrm{M}\,\beta + \mathrm{N}, \qquad \gamma' = \mathrm{M}'\beta + \mathrm{N}',$$

where K, L, M, N, K′, L′, M′, N′, are known functions of the constant tangential coordinates of the four given right lines. Substituting those values in (b), (b′), we shall have

$$(\mathrm{K}\,\xi + v + \mathrm{M})\,\beta + (\mathrm{L}\,\xi + \mathrm{N}) = 0,$$
$$(\mathrm{K}'v + \xi + \mathrm{M}')\,\beta + (\mathrm{L}'v + \mathrm{N}') = 0;$$

eliminating $\beta$ from those equations, we get the tangential equation of a conic section.

When the point chosen is one of the angles of the quadrilateral the section becomes a point.

*Let a series of concentric conic sections, having the same* MINOR DIRECTRICES,* *be cut by a right line, the segments of this transversal made by any pair of such sections, subtend equal angles at the centre.*

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* Let a right line be drawn parallel to the major axis of an ellipse at the distance $\frac{b}{e}$ from the centre; $e$ denoting the eccentricity of the section; and a point assumed on the minor axe at the distance $b\,e$ from the centre, this right

Let $x\xi + y\upsilon = 1$ be the projective equation of the transversal, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, that of the section let $y = x\tau$; eliminating $x, y$, from these three equations we obtain

$$a^2(1 - b^2\upsilon^2)\tau^2 - 2a^2 b^2 \xi\upsilon . \tau + b^2(1 - a^2\xi^2) = 0.$$

Let $\tau'$, $\tau''$ be the roots of this equation, then

$$\tau' + \tau'' = \frac{2b^2\xi\upsilon}{1 - b^2\upsilon^2}, \qquad \tau'\tau'' = \frac{b^2(1 - a^2\xi^2)}{a^2(1 - b^2\upsilon^2)}.$$

$\chi$ denoting the sum of the angles which the diameters passing through the points where the right line meets the section, make with the transverse axe, we shall have

$$\tan\chi = \frac{\tau' + \tau''}{1 - \tau\tau'} = \frac{2\xi\upsilon}{\left(\frac{1}{b^2} - \frac{1}{a^2}\right) - \left(\xi^2 + \upsilon^2\right)};$$

let $k$ denote the distance of one of the minor directrices from the centre, then

$$\frac{1}{b^2} - \frac{1}{a^2} = \frac{1}{k^2},$$

and

$$\tan\chi = \frac{2k^2\xi\upsilon}{1 - k^2(\xi^2 + \upsilon^2)},$$

which value of $\tan\chi$ is independent of the values of the axes of the assumed section, and depends only on the position of the transversal and the distance of the minor directrix.

*Two cords being drawn through two fixed points on a conic*

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line and point may be termed for the sake of distinction, the *minor directrix* and *minor focus* of the section.

In the hyperbola, the minor directrices are parallel to the imaginary, and the minor foci are placed on the real or transverse axe.

In the parabola they are infinitely distant; in the circle the minor focus coincides with the centre, and the minor directrix is infinitely distant.

In the equilateral hyperbola the two species of foci and directrices coincide, this accounts for the centre of the equilateral hyperbola possessing focal properties as has been long known.

To conclude, the minor directrices and foci will be found to possess as many curious and elegant properties as the common directrices and foci of the central conic sections.

*through a third point arbitrarily assumed on the section, will intercept on the minor directrix a segment which subtends at the centre a constant angle.*

When the fixed points are the extremities of the minor axe in the ellipse, or of the transverse axe in the hyperbola the constant angle is a right angle.

When the ellipse degenerates into a circle, the minor directrix recedes to infinity, hence the diameters drawn to the extremities of the segment of the minor directrix intercepted by the cords, become parallel to the cords, or in other words, the angle in a segment of a circle is constant, and the angle in a semicircle is a right angle.

## IV.

### ON THE PARABOLA.

By a similar analysis it may be shown, that the tangential equation of a parabola is

$$\left.\begin{aligned}(p\cos\theta + l)\,\xi^2 + (q\sin\theta + l)\,v^2 + (p\sin\theta + q\cos\theta)\,\xi\,v \\ + \cos\theta\,\xi + \sin\theta\,v = 0,\end{aligned}\right\} \quad (13)$$

where $l =$ one-fourth of the parameter, $\theta$ the inclination of the axis of the section to the axis of $x$, $p$ and $q$ the projective coordinates of the focus.

Hence the general equation of the parabola being

$$f\xi^2 + f'v^2 + g\,\xi\,v + h\,\xi + h'\,v = 0, \quad (14)$$

we shall have the following equations,

$$\tan\theta = \frac{h'}{h}, \quad (15) \qquad l = \frac{fh'^2 + f'h^2 - ghh'}{(h^2 + h'^2)^{\frac{3}{2}}}, \quad (16)$$

$$p = \frac{gh' + h(f - f')}{h^2 + h'^2}, \qquad q = \frac{gh - h'(f - f')}{h^2 + h'^2}, \quad (17)$$

which completely determine the magnitude and position of the parabola.

In the tangential equation of the parabola the absolute term $= 0$; but in the projective equation of a conic section,

$$\text{A}\,x^2 + \text{A}'\,y^2 + 2\text{B}\,xy + 2\text{C}\,x + 2\text{C}'\,y = 1,$$

the relation $\text{AA}' - \text{B}^2 = 0$ indicates that the curve is a parabola, and the evanescence of the absolute term that the origin is on the curve, while in the tangential equation, the latter relation holds when the curve is a parabola, and the former shows, page 4, that the origin of coordinates is on the curve.

*An angle of given magnitude moves along a given right line, one side passes through a fixed point, the other envelopes a parabola.*

Let the given line and the perpendicular on it from the given point be taken as axes of coordinates; $a =$ distance of point from fixed line, $n =$ tangent of given angle, then

$$\frac{\xi}{v} = \frac{n + \frac{1}{av}}{1 - \frac{n}{av}}, \text{ or reducing}$$

$$n a v^2 - a \xi v + n \xi + v = 0, \qquad \text{(a)}$$

the tangential equation of a parabola, as the absolute term is zero. Comparing the terms of this equation with those of (14), we have

$$f = 0, \quad f' = na, \quad g = -a, \quad h = n, \quad h' = 1,$$

hence

$$\tan\theta = \frac{1}{n}, \qquad l = \frac{na}{\sqrt{1+n^2}}, \qquad = a\cos\theta, \qquad p = -a, \qquad q = 0.$$

*A series of parabolas are inscribed in a triangle, the locus of the foci is the circumscribing circle.*

Let the base of the triangle be taken as axis of $x$, the origin being placed at an angle of the triangle, $n$ the tangent of the angle which the second side passing through the origin makes with the axis of $x$, then as the base is a tangent, $f' = 0$ and as the second side is a tangent $g = nf$ $\because$ the equation of one of the parabolas is

$$f\xi^2 + nf\xi v + h\xi + h'v = 0; \qquad \text{(a)}$$

let the tangential equation of the third side of the triangle be $\xi = \frac{1}{a}$, $v = \frac{1}{b}$, substituting those values in (a) we shall have the equation

$$fb + nfa + hab + h'a^2 = 0. \qquad \text{(b)}$$

Now solving equations (17) for $h$ and $h'$, first putting $f' = 0$, $g = nf$, we shall have

$$h = \frac{f(p + nq)}{p^2 + q^2}; \qquad h' = \frac{f(np - q)}{p^2 + q^2}.$$

Substituting those values in (b) and dividing by $f$ we get

$$p^2 + q^2 + ap + \frac{(nb - a)}{na + b} q = 0,$$

the equation of a circle circumscribing the triangle.*

* This question has been solved by Mr. Lubbock in the Philosophical Magazine for August, 1836, page 100, where the solution by the ordinary methods occupies nearly five pages.

It is almost needless to observe, that analogous formulæ may be established by reasonings precisely similar for oblique coordinates; an example of the application of such formulæ may suffice.

*A series of parabolas are inscribed in the same triangle, the lines joining the points of contact of each parabola with the opposite vertices of the triangle meet in a point; the locus of this point is the minimum ellipse circumscribing the triangle.*

Let the sides $a$, $b$ of the triangle be taken as axes of coordinates, the equation of the parabola is

$$g\,\xi\,v + h\,\xi + h' v = 0, \tag{a}$$

and as the third side of the triangle whose equations are $\xi = \frac{1}{a}$, $v = \frac{1}{b}$ is a tangent to the parabola

$$g + h\,b + h'\,a = 0, \tag{b}$$

the parabola touches the axis of $x$ at a distance from the origin $= \frac{-g}{h'}$ and the axis of $y$ at the distance $\frac{-g}{h}$: the projective equations of the lines joining these points with the opposite vertices are

$$y - b = \frac{b\,h'x}{g}, \quad (c) \qquad x - a = \frac{a\,h\,y}{g}; \tag{d}$$

and these equations combined give the projective equations of the point required, solving those equations for $h$, $h'$, and substituting their values in (b) we get

$$a^2 y^2 + b^2 x^2 + abxy - a^2by - ab^2x = 0, \tag{e}$$

the equation of an ellipse circumscribing the triangle; let the origin be translated to the centre of gravity of the triangle the axes remaining parallel, and equation (e) is transformed into

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{xy}{ab} = \frac{1}{3}.$$

*The vertical angle, and the sum of the sides of a triangle are constant, the base envelopes a parabola.*

Taking the sides as axes of coordinates, let the sum be equal to $a$, then by the question $\frac{1}{\xi} + \frac{1}{v} = a$, or

$$a\,\xi\,v - \xi - v = 0,$$

the tangential equation of a parabola.

# CHAPTER II.

## I.

### ON THE TANGENTIAL EQUATIONS OF A POINT, RIGHT LINE, AND PLANE.

LET $\alpha$, $\beta$, $\gamma$, be the projective coordinates of a point, the tangential equation of this point is

$$\alpha \xi + \beta v + \gamma \zeta = 1. \tag{18}$$

The tangential equations of a right line, whose projective equations are $x = mz + a$, $y = nz + b$, are

$$m \xi + n v + \zeta = 0; \qquad a \xi + b v - 1 = 0. \tag{19}$$

The tangential equations of a plane are

$$\xi = \text{constant}, \qquad v = \text{constant}, \qquad \zeta = \text{constant}. \tag{20}$$

## II.

### ON THE TRANSFORMATION OF COORDINATES.

Let three rectangular axes OX, OY, OZ, be drawn through a fixed point O, meeting a given plane in three points; let the reciprocals of the distances of those points from the origin be denoted by $\xi$, $v$, $\zeta$, the reciprocals of the distances of the corresponding points for three other rectangular axes passing through the same origin by $\xi'$, $v'$, $\zeta'$, let the axis of X$'$ make with the original axes the angles $\lambda$, $\mu$, $\nu$, the axes of Y$'$ and Z$'$ with the same axes the angles $\lambda' \mu' \nu'$ and $\lambda'' \mu'' \nu''$ respectively, then it may be easily shown that

$$\left.\begin{aligned} \xi &= \xi' \cos\lambda + v' \cos\lambda' + \zeta' \cos\lambda'', \\ v &= \xi' \cos\mu + v' \cos\mu' + \zeta' \cos\mu'', \\ \zeta &= \xi' \cos\nu + v' \cos\nu' + \zeta' \cos\nu''. \end{aligned}\right\} \tag{21}$$

The axes remaining parallel, let the origin be translated to a point, whose projective coordinates are $p$, $q$, $r$, then

$$\left.\begin{array}{c}\xi = \dfrac{\xi'}{1+p\xi'+qv'+r\zeta'}, \qquad v = \dfrac{v'}{1+p\xi'+qv'+r\zeta'}, \\ \zeta = \dfrac{\zeta'}{1+p\xi'+qv'+r\zeta'}.\end{array}\right\} \quad (22)$$

## III.

### ON THE TANGENTIAL EQUATIONS OF SURFACES OF THE SECOND ORDER.

It may be easily shown, that the tangential equation of a surface of the second order referred to its centre and axes is

$$a^2\xi^2 + b^2 v^2 + c^2\zeta^2 = 1. \qquad \text{(a)}$$

The surface being now referred to three rectangular axes passing through the centre by the help of formulæ (21) the equation (a) assumes the form

$$\text{R}\,\xi^2 + \text{R}'\,v^2 + \text{R}''\,\zeta^2 + 2\text{Q}\,\zeta\,v + 2\text{Q}'\,\xi\,\zeta + 2\text{Q}''\,\xi\,v = 1; \quad (23)$$

R, R′, R″, Q, Q′, Q″, being functions of the three axes and nine angles; hence the general equation of a surface of the second order referred to three rectangular axes passing through the centre being

$$a\,\xi^2 + a'\,v^2 + a''\,\zeta^2 + 2\beta\,\zeta\,v + 2\beta'\,\xi\,\zeta + 2\beta''\,\xi\,v = 1; \quad (24)$$

we shall have to determine the twelve unknown quantities, namely, the nine angles and the squares of the three semiaxes; twelve equations, six of which are given by the known relations between the nine angles, and six by equating, term by term, the coefficients of equations (23) and (24), which completely solve the problem. It is obvious, that the solution of those twelve equations would lead to very complicated and unmanageable expressions; but the difficulty may be eluded from other considerations.

Conceive a sphere concentric to the given surface, whose radius may be equal to one of the semi-axes of the surface; let a common tangent plane be drawn to the sphere and surface; it is clear that if the tangent plane touches the surface at the extremity of the axe which is equal to the radius of the sphere, that the point of contact of the tangent plane with the surface will be also on the surface of the sphere; let $(x'y'z')$ be the projective coordinates of this point, $\rho =$ square of the radius of the sphere, now it may be proved that if the general tangential equation of the surface is written in the form

$$(a\xi + \beta'\zeta + \beta''v)\xi + (a'v + \beta''\xi + \beta\zeta)v + (a''\zeta + \beta v + \beta'\xi)\zeta = 1;$$

that the three coordinates of the point of contact of the tangent plane with the surface are given by the equations

$$\left.\begin{aligned} a\xi + \beta'\zeta + \beta''v &= x; \\ a'v + \beta''\xi + \beta\zeta &= y; \\ a''\zeta + \beta v + \beta'\xi &= z. \end{aligned}\right\} \quad (25)$$

But this point is supposed to be also on the surface of the sphere, hence

$$\rho\xi = x, \qquad \rho v = y, \qquad \rho\zeta = z.$$

Subtracting these equations from the former, we shall have

$$\begin{aligned} (a-\rho)\xi + \beta'\zeta + \beta''v &= 0, \\ (a'-\rho)v + \beta''\xi + \beta\zeta &= 0, \\ (a''-\rho)\zeta + \beta v + \beta'\xi &= 0. \end{aligned}$$

Eliminating from these equations $\xi$, $v$, $\zeta$, we have to determine the axes, the cubic equation

$$\left.\begin{aligned} (a-\rho)(a'-\rho)(a''-\rho) - \beta^2(a-\rho) - \beta'^2(a'-\rho) - \beta''^2(a''-\rho) \\ +2\beta\beta'\beta'' = 0. \end{aligned}\right\} \quad (26)$$

Let the three roots of this equation be $\tau^2$, $\tau'^2$, $\tau''^2$, put $(a-\tau^2) = \delta$, $(a'-\tau^2) = \delta'$, $(a''-\tau^2) = \delta''$, let $\lambda$, $\mu$, $\nu$, be the angles which the axis $\tau$ makes with the axes of coordinates, then

$$\left.\begin{aligned} \cos^2\lambda &= \frac{(\delta'\delta''-\beta^2)}{[(\delta'\delta''-\beta^2)+(\delta''\delta-\beta'^2)+(\delta\delta'-\beta''^2)]}, \\ \cos^2\mu &= \frac{(\delta''\delta-\beta'^2)}{[(\delta'\delta''-\beta^2)+(\delta''\delta-\beta'^2)+(\delta\delta'-\beta''^2)]}, \\ \cos^2\nu &= \frac{(\delta\delta'-\beta''^2)}{[(\delta'\delta''-\beta^2)+(\delta''\delta-\beta'^2)+(\delta\delta'-\beta''^2)]}. \end{aligned}\right\} \quad (27)^*$$

In the same manner may the angles which the axes $\tau'$, $\tau''$, make with the axes of coordinates be discovered, hence the twelve unknown quantities are determined.

Should $\tau$ be one of the equal axes of a surface of revolution, we shall have the relations

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* It is plain, that similar symmetrical formulæ may be given in the analogous problems of finding the position of the axes of a surface of the second order, by the common method of projective coordinates, and of the three principal axes of rotation of a homogeneous body.

$$\delta'\delta'' - \beta^2 = 0, \quad \delta''\delta - \beta'^2 = 0, \quad \delta\,\delta' - \beta''^2 = 0; \qquad (28)$$

for these three relations satisfy the cubic equation (26) and its first derivative, which is

$$[(\alpha-\rho)(\alpha'-\rho)-\beta''^2]+[(\alpha'-\rho)(\alpha''-\rho)-\beta^2]+[(\alpha''-\rho)(\alpha-\rho)-\beta'^2]=0,$$

hence (26) must have two equal roots.

The equation of a surface referred to three rectangular axes passing through the centre being

$$\text{R}\,\xi^2 + \text{R}'\,v^2 + \text{R}''\,\zeta^2 + 2\text{Q}\,\zeta\,v + 2\text{Q}'\,\zeta\,\xi + 2\text{Q}''\,\xi\,v = 1; \qquad \text{(a)}$$

let the origin be translated to a point whose projective coordinates are $p$, $q$, $r$, and equation (a) is transformed into

$$\left.\begin{array}{l}(\text{R}-p^2)\,\xi^2+(\text{R}'-q^2)v^2+(\text{R}''-r^2)\zeta^2+2\,(\text{Q}-qr)\,\zeta v+2\,(\text{Q}'-pr)\,\xi\zeta \\ \qquad +2(\text{Q}''-pq)\,\xi v-2p\xi-2qv-2r\zeta=1.\end{array}\right\} \quad \text{(b)}$$

Hence the general tangential equation of a surface referred to any rectangular axes being

$$\left.\begin{array}{l}\alpha\,\xi^2 + \alpha'\,v^2 + \alpha''\,\zeta^2 + 2\beta\,\zeta\,v + 2\beta'\,\xi\,\zeta + 2\beta''\,\xi\,v \\ \qquad + 2\gamma\,\xi + 2\gamma'\,v + 2\gamma''\,\zeta = 1;\end{array}\right\} \quad (29)$$

$\gamma$, $\gamma'$, $\gamma''$, denote the projective coordinates of the centre.

The general equation (29), the origin being translated to the centre, becomes

$$\left.\begin{array}{l}(\alpha + \gamma^2)\,\xi^2 + (\alpha' + \gamma'^2)\,v^2 + (\alpha'' + \gamma''^2)\,\zeta^2 \\ +2(\beta + \gamma'\gamma'')v\zeta + 2(\beta' + \gamma\gamma'')\xi\zeta + 2(\beta'' + \gamma\gamma')\xi v = 1.\end{array}\right\} (30)$$

The polar plane of the origin is given by the three simultaneous equations

$$\left.\begin{array}{l}\alpha\,\xi + \beta'\zeta + \beta''v + \gamma = 0; \\ \alpha'v + \beta''\,\xi + \beta\,\zeta + \gamma' = 0; \\ \alpha''\,\zeta + \beta\,v + \beta'\,\xi + \gamma'' = 0;\end{array}\right\} \quad (31)$$

which severally denote the tangential equations of the poles of the planes of $yz$, $xz$, and $xy$.

The equation of condition

$$\alpha\,\beta^2 + \alpha'\,\beta'^2 + \alpha''\,\beta''^2 = \alpha\,\alpha'\,\alpha'' + 2\beta\,\beta'\,\beta'', \qquad (32)$$

denotes that the origin is on the surface.*

When $\alpha = \alpha' = \alpha''$ and $\beta = 0$, $\beta' = 0$, $\beta'' = 0$, in the general

* The reciprocal analogies between the equations just given, and the similar ones in projective coordinates are very obvious and striking.

equation (29) the surface is one of revolution the origin being at a focus; for if the origin be translated to the centre, and the above condition introduced, the cubic equation (26) is transformed into

$$\left.\begin{array}{c}\rho^3-\rho^2[3a+\gamma^2+\gamma'^2+\gamma''^2]+\rho[3a^2+2a(\gamma^2+\gamma'^2+\gamma''^2)] \\ -[a^3+a^2(\gamma^2+\gamma'^2+\gamma''^2)]=0;\end{array}\right\}$$

let $a+\gamma^2+\gamma'^2+\gamma''^2=k^2$, and this equation may be changed into

$$\rho^3-\rho^2(2a+k^2)+\rho(a^2+2ak^2)-a^2k^2=0,$$

of which the three roots are manifestly $a$, $a$, and $k^2$.

Putting now $k$ for $\tau$ in the expressions (27) we get

$$\left.\begin{array}{c}\cos\lambda=\dfrac{\gamma}{\sqrt{\gamma^2+\gamma'^2+\gamma''^2}},\qquad \cos\mu=\dfrac{\gamma'}{\sqrt{\gamma^2+\gamma'^2+\gamma''^2}}, \\ \cos\nu=\dfrac{\gamma''}{\sqrt{\gamma^2+\gamma'^2+\gamma''^2}};\end{array}\right\} \quad (33)$$

which equations determine the position of the semi-axis $k$, but the line drawn from the centre to the origin makes the same angles with the axes of coordinates $\because$ this line coincides with the semi-axis $k$, hence the origin is on this axis and $k^2-a=\gamma^2+\gamma'^2+\gamma''^2$; $\because$ the eccentric distance is $=$ to the distance of the origin from the centre, or the focus is at the origin.

A cone whose vertex is $(x'\,y'\,z')$ enveloping a surface of the second order, whose tangential equation is $\Phi(\xi,v,\zeta)=0$, is given by the simultaneous equations $\Phi(\xi,v,\zeta)=0$, $x'\xi+y'v+z'\zeta=1$.

The curve of contact of a cone with the enveloped surface of the second order $\Phi(\xi,v,\zeta)=0$ is a plane curve.

Let $(x,y,z)$ be the projective coordinates of any point on the curve, then, see (25), as this point is on the tangent plane, we shall have the four linear equations, the origin being at the centre:

$$\left.\begin{array}{l}a\xi+\beta'\zeta+\beta''v-x=0, \\ a'v+\beta''\xi+\beta\zeta-y=0, \\ a''\zeta+\beta v+\beta'\xi-z=0, \\ x'\xi+y'v+z'\zeta-1=0.\end{array}\right\} \quad (34)$$

Now of the six linear quantities $x$, $y$, $z$, $\xi$, $v$, $\zeta$, in these four equations eliminating the three latter, we shall have a linear equation connecting the three former

$$Px+Qy+Rz=1 \qquad \text{(f)}$$

the projective equation of a plane.

Write the latter equation (f) in the form

$$\xi' x + v' y + \zeta' z = 1, \tag{e}$$

and solve the first three of the group (34) for $\xi, v, \zeta,$ then

$$\xi = \mathrm{L}x + \mathrm{M}y + \mathrm{N}z, \quad v = \mathrm{L}'x + \mathrm{M}'y + \mathrm{N}'z, \quad \zeta = \mathrm{L}''x + \mathrm{M}''y + \mathrm{N}''z.$$

Multiply these equations by $x', y', z'$, respectively, and add

$$\left.\begin{array}{l}(\mathrm{L}x' + \mathrm{L}'y' + \mathrm{L}''z')x + (\mathrm{M}x' + \mathrm{M}'y' + \mathrm{M}''z')y + (\mathrm{N}x' + \mathrm{N}'y' + \mathrm{N}''z')z \\ \qquad = x'\xi + y'v + z'\zeta.\end{array}\right\} \tag{k}$$

Now by (34) $x'\xi + y'v + z'\zeta = 1$, hence equation (k) is changed into

$$(\mathrm{L}x' + \mathrm{L}'y' + \mathrm{L}''z')x + (\mathrm{M}x' + \mathrm{M}'y' + \mathrm{M}''z')y + (\mathrm{N}x' + \mathrm{N}'y' + \mathrm{N}''z')z = 1; \tag{m}$$

but this equation is identical with (e), therefore,

$$\left.\begin{array}{l}\xi' = \mathrm{L}x' + \mathrm{L}'y' + \mathrm{L}''z'; \\ v' = \mathrm{M}x' + \mathrm{M}'y' + \mathrm{M}''z'; \\ \zeta' = \mathrm{N}x' + \mathrm{N}'y' + \mathrm{N}''z';\end{array}\right\} \tag{35}$$

which equations exhibit the relations between the projective coordinates of the pole, and the tangential coordinates of the polar plane.

Put

$$\begin{array}{l}\mathrm{L}x' + \mathrm{L}'y' + \mathrm{L}''z' = \mathrm{X}, \\ \mathrm{M}x' + \mathrm{M}'y' + \mathrm{M}''z' = \mathrm{Y}, \\ \mathrm{N}x' + \mathrm{N}'y' + \mathrm{N}''z' = \mathrm{Z};\end{array}$$

hence we shall have

$$\xi = \mathrm{X}; \quad v = \mathrm{Y}; \quad \zeta = \mathrm{Z}; \tag{36}$$

omitting the traits as no longer necessary.

From these equations the whole theory of poles and polars follows with singular facility. Thus, if the polar plane is fixed; $\mathrm{X} =$ constant, $\mathrm{Y} =$ constant, and $\mathrm{Z} =$ constant, which three equations determine the projective coordinates of the pole.

When $\xi, v, \zeta$, are connected by two linear equations, so also are X, Y, Z, or if the polar plane passes through a fixed right line, the pole also traverses a right line.

When $\xi, v, \zeta$, are connected by one linear equation, the same relation exists between X, Y, Z, or if the polar plane passes through a fixed point, the pole traverses a fixed plane.

When $\xi, v, \zeta$, are related by an equation of the second degree, so also are X, Y, Z, or if the polar plane envelopes a surface of the second order, the pole traverses a surface of the same degree.

Generally, *if the tangential equation of any surface be* $\Phi(\xi, \upsilon, \zeta) = 0$, *the projective equation of its* RECIPROCAL POLAR* is $\Phi(X, Y, Z) = 0$.

Should for simplicity, the directrix surface be a sphere, of radius equal to unity, *the tangential equation of any surface being* $\Phi(\xi, \upsilon, \zeta) = 0$, *the projective equation of its reciprocal polar will be* $\Phi(x, y, z) = 0$, *and conversely.*

By the aid of this very remarkable theorem, and of the properties of tangential equations already discussed, we may reduce the whole theory of reciprocal polars under the dominion of analysis with the greatest ease; the following are a few of the most obvious subordinate relations deducible from this fundamental theorem.

Given the projective or tangential equations of any surface $F(x, y, z,) = 0$, or $\Phi(\xi, \upsilon, \zeta,) = 0$, we may write down the tangential or projective equations of its reciprocal polar, $F(\xi, \upsilon, \zeta,) = 0$, or $\Phi(x, y, z,) = 0$, from mere inspection.

Conceive a figure composed of points, right lines, planes, curves of single and double curvature, and curved surfaces; a surface of the second order being described as directrix. Imagine another figure constructed, whose points, right lines, and planes, shall be the poles, conjugate polars, and polar planes, of the planes, right lines, and points, of the original figure; these two figures may be called *reciprocal polars*, one of the other.

From the reciprocal relations between the two equations $\Phi(\xi, \upsilon, \zeta,) = 0$, and $\Phi(x, y, z,) = 0$, we may conclude that if—

*In one figure, a surface passes through* n *given points, in the other we shall have a surface touching* n *given planes.*

*In one figure, a group of right lines being situated in the same plane, in the other we shall have as many right lines meeting in a point.*

*In the one, a system of points being given on a right line, in the other we shall have as many planes, cutting in the same right line.*

*In the one, a system of points being given on a plane, in the other we shall have as many planes, passing through the same fixed point.*

*In the one, the points of intersection of a plane curve with a right*

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* Let a point be assumed on a surface (S), and the polar plane of this point taken relative to a surface of the second order (C); as the assumed point varies on the surface (S); the polar plane envelopes a surface ($\Sigma$), which is termed the *reciprocal polar* of the given surface (S).—*Annales de Mathématiques*, tom. viii. page 201.

*line being given, in the other we shall have as many tangent planes to the same conic surface, all meeting in a right line passing through the vertex.*

*In the one, a number of points being common to two or more plane curves, situated in the same plane; in the other we shall have as many tangent planes to two or more cones having the same vertex.*

*In the one, a system of points being given on the same curved surface; in the other we shall have an equal number of tangent planes to its reciprocal surface.*

*In the one, the intersection of a surface with a plane being given; in the other we shall have a cone circumscribing the reciprocal surface.*

*In the one, the points where a curved surface is met by a right line being given; in the other, we shall have as many tangent planes to the reciprocal surface, cutting in the same right line.*

*In the one, points being common to three or more curved surfaces, we shall have as many tangent planes in the other, common to as many reciprocal surfaces.*

*In the one, a system of points being given on a curve of double curvature, so many planes will be in the other, tangents to the same developable surface.*

*In the one, the points of intersection of a plane with a curve of double curvature being given; in the other, so many tangent planes to the same developable surface pass through the same point.*

*In the one, points being common to two or more curves of double curvature; in the other, so many tangent planes exist to two or more developable surfaces.*

*In the one, if a series of tangent planes may be drawn to a curved surface through a point* on *it; in the other, we shall have as many points of contact of a curved surface with a tangent plane. Hence, if a cusp is found on the one, a curve of plane contact exists on the other.**

As the reciprocal polar of a surface of the second order is also a surface of the same degree, a great variety of the properties of such surfaces may be deduced in this manner; for example, it may be easily shown, that if *two surfaces of the second order intersect in a plane curve, they must again intersect in another plane curve;* hence, *if*

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* This is a known relation in the wave theory of light, between the wave-surface and its reciprocal polar, the surface of *wave-slowness*, or of *refraction*, of Professors Sir William Hamilton and Mac Cullagh.

*two such surfaces are enveloped by one cone, they are also enveloped by a second.* And thus a duality exists between the properties of curves and surfaces of the second degree, which in the general case is found only between curves and surfaces, and their reciprocal polars; again—

If one surface is generated by rectilinear generatrices, so also will its reciprocal polar; hence, of surfaces of the second order, the *hyperbolic paraboloid*, and the *continuous hyperboloid*, or that of one sheet, are exclusively reciprocal polars of themselves or of each other, whence it follows, that the properties of those surfaces are reciprocal to each other.

EXAMPLES.

I. *If a series of planes retrench from a cone of the second degree a constant volume, they will envelope a discontinuous hyperboloid, or one of two sheets.*

In the first place it may be shown, that if

$$\mathrm{A}\,x^2 + \mathrm{A}'\,y^2 + 2\,\mathrm{B}\,x\,y + 2\,\mathrm{C}\,x + 2\,\mathrm{C}'y = 1,$$

be the projective equation of a conic section, its area

$$= \frac{\pi\,[(\mathrm{A} + \mathrm{C}^2)\,(\mathrm{A}' + \mathrm{C}'^2) - (\mathrm{B} + \mathrm{CC}')^2]}{(\mathrm{AA}' - \mathrm{B}^2)^{\frac{3}{2}}}, \qquad \text{(a)}$$

let

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0, \qquad \text{(b)}$$

be the equation of the cone, and

$$\xi\,x + v\,y + \zeta\,z = 1, \qquad \text{(c)}$$

that of the secant plane. Eliminating $z$ between the equations (b) and (c) we shall have

$$\left(\frac{c^2\zeta^2}{a^2} - \xi^2\right)x^2 + \left(\frac{c^2\zeta^2}{b^2} - v^2\right)y^2 - 2\xi v\,x\,y + 2\xi x + 2v\,y = 1, \qquad \text{(d)}$$

the equation of the projection on the plane of $x\,y$, of the section of the cone (b) by the secant plane (c).

Let $\mathrm{S}$ = area of the section (d), $\theta$ the angle between the plane of $x\,y$ and the plane (c), then $\mathrm{S}\sec\theta$ = area of section of cone made by the plane (c) and $\dfrac{\cos\theta}{\zeta}$ = perpendicular from the origin on this plane,

$\therefore$ the volume $= \dfrac{\mathrm{S}}{3\,\zeta} = \dfrac{\pi\,a\,b\,c}{3}$ suppose,

Now substituting the coefficients of (d) in (a) we obtain

$$\frac{\mathrm{s}}{\zeta} = \frac{\pi\, abc}{(c^2\zeta^2 - a^2\xi^2 - b^2 v^2)^{\frac{1}{2}}},$$

hence reducing we find

$$c^2\zeta^2 - a^2\xi^2 - b^2 v^2 = 1,$$

the tangential equation of a discontinuous hyperboloid.

In nearly the same way may it be shown, *that if a series of planes retrench from any surface of the second order a constant volume; the enveloped surface is a surface concentric, similar, and similarly situated.*

II. *A series of surfaces of the second order touch seven fixed planes; the poles of any given plane relative to those surfaces, are also on a fixed plane.*

Let the given plane be taken as that of $xy$, and let the equation of one of the surfaces be

$$\left.\begin{aligned} a\xi^2 + a' v^2 + a''\zeta^2 + 2\beta v\zeta + 2\beta'\xi\zeta + 2\beta''\xi v \\ + 2\gamma\xi + 2\gamma' v + 2\gamma''\zeta = 1; \end{aligned}\right\} \quad \text{(a)}$$

the tangential equation of the pole of the plane of $xy$ relative to this surface is

$$a''\zeta + \beta v + \beta'\xi + \gamma'' = 0. \quad \text{See (31).} \qquad \text{(b)}$$

Now as there are seven linear equations to determine the nine unknown coefficients of (a), we may eliminate any six, and connect the three remaining by an equation which will be also linear. Eliminating then $a, a', \beta'', \gamma, \gamma', \gamma''$, and $a, a', a'', \beta'', \gamma, \gamma'$, successively, we obtain

$$a'' = \mathrm{L}\beta + \mathrm{M}\beta' + \mathrm{N},$$
$$\gamma'' = \mathrm{L}'\beta + \mathrm{M}'\beta' + \mathrm{N}';$$

$\mathrm{L}, \mathrm{M}, \mathrm{N}, \mathrm{L}', \mathrm{M}', \mathrm{N}'$, being determinate functions of the twenty-one constant tangential coordinates of the seven fixed planes.

Substituting those values in (b) we get the equation

$$(\mathrm{L}\zeta + v + \mathrm{L}')\beta + (\mathrm{M}\zeta + \xi + \mathrm{M}')\beta' + (\mathrm{N}\zeta + \mathrm{N}') = 0. \qquad \text{(d)}$$

Now this equation is satisfied, leaving $\beta$ and $\beta'$ indeterminate, by putting each of the three factors in (d) = zero; solving those equations we find

$$\xi = \text{constant}, \qquad v = \text{constant}, \qquad \zeta = \text{constant};$$

the three tangential equations of a fixed plane.

When there are eight fixed planes, it may be similarly shown, *that the locus of the poles of any given plane relative to those surfaces is a right line.*

III. *A tangent plane is drawn to three concyclic surfaces* of the second order, the three points of contact, two by two, subtend right angles at the centre.*

Let

$$a^2\xi^2 + b^2 v^2 + c^2\zeta^2 = 1, \text{ and } a'^2\xi^2 + b'^2 v^2 + c'^2\zeta^2 = 1, \qquad \text{(a)}$$

be the tangential equations of two concyclic surfaces of the second order.

Subtracting those equations one from the other, we obtain

$$(a^2 - a'^2)\,\xi^2 + (b^2 - b'^2)\,v^2 + (c^2 - c'^2)\,\zeta^2 = 0; \qquad \text{(b)}$$

but as the surfaces are concyclic,

$$\frac{1}{a'^2} = \frac{1}{a^2} + \frac{1}{k^2}; \quad \frac{1}{b'^2} = \frac{1}{b^2} + \frac{1}{k^2}; \quad \frac{1}{c'^2} = \frac{1}{c^2} + \frac{1}{k^2}.$$

Substituting those values of $a'$, $b'$, $c'$, in (b), we find

$$a^2 a'^2 \xi^2 + b^2 b'^2 v^2 + c^2 c'^2 \zeta^2 = 0. \qquad \text{(d)}$$

Now it has been shown, page 13, that if $(x, y, z,)$ are the coordinates of the point of contact of a tangent plane,

$$x = a^2\xi, \quad y = b^2 v, \quad z = c^2\zeta;$$

and if $\lambda$, $\mu$, $\nu$, are the angles which the diameter through the point of contact makes with the axes,

$$\cos\lambda = \frac{x}{r} = \frac{a^2\xi}{r}, \quad \cos\mu = \frac{b^2 v}{r}, \quad \cos\nu = \frac{c^2\zeta}{r},$$

similarly

$$\cos\lambda' = \frac{a'^2\xi}{r'}, \quad \cos\mu' = \frac{b'^2 v}{r'}, \quad \cos\nu' = \frac{c'^2\zeta}{r'}.$$

Substituting those values in (d) we obtain

$$r r' (\cos\lambda \cos\lambda' + \cos\mu \cos\mu' + \cos\nu \cos\nu') = 0.$$

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* It is strange how the properties of systems of concentric surfaces of the second order, having coincident circular sections, or as they may be more briefly termed *concyclic surfaces*, have hitherto almost wholly, at least sofar as the author is aware, escaped the observation of geometers; it is the more remarkable as the theorems connected with the subject are numerous, and many of great elegance.

So far indeed as the properties of such cones are concerned, and of spherical conics thence derived, M. Chasles has discussed them in two memoirs of singular simplicity and beauty, published in the *Brussels Transactions*.

IV. *The polar planes of a fixed point, relative to a series of concyclic surfaces of the second order, pass through a fixed right line.*

*And the diametral plane passing through this line, is perpendicular to the diameter passing through the given point.*

Let

$$a^2 \xi'^2 + b^2 v'^2 + c^2 \zeta'^2 = 1, \tag{a}$$

be the tangential equation of one of the concyclic surfaces, $\xi$, $v$, $\zeta$, the tangential coordinates of the polar plane of the fixed point, (whose projective coordinates are $p$, $q$, $r$,) relative to this surface, then

$$p = a^2 \xi, \qquad q = b^2 v, \qquad r = c^2 \zeta, \tag{b}$$

see (35); but as the surfaces are concyclic,

$$\frac{1}{b^2} = \frac{1}{a^2} + \frac{1}{h^2}, \qquad \frac{1}{c^2} = \frac{1}{a^2} + \frac{1}{h'^2}, \tag{c}$$

eliminating $a$, $b$, $c$, from the five equations (b), (c), we shall have the two resulting equations

$$\frac{v}{q} = \frac{\xi}{p} + \frac{1}{h^2}, \quad \text{and} \quad \frac{\zeta}{r} = \frac{\xi}{p} + \frac{1}{h'^2}; \tag{d}$$

the two tangential equations of a right line, which is manifestly fixed, being independent of the semi-axes $a$, $b$, $c$, of the surface.

Again, as the diametral plane passing through the fixed line (d), passes through the origin, $\frac{1}{\xi} = 0$, $\frac{1}{v} = 0$, and $\frac{1}{\zeta} = 0$, putting those values in equations (d) we find $\frac{v}{\xi} = \frac{q}{p}$; and $\frac{\zeta}{\xi} = \frac{r}{p}$; let $\lambda$, $\mu$, $\nu$, be the angles which a diameter normal to this plane makes with the axes, then $\frac{\cos\mu}{\cos\lambda} = \frac{v}{\xi} = \frac{q}{p}$, and $\frac{\cos\nu}{\cos\lambda} = \frac{\zeta}{\xi} = \frac{r}{p}$, hence

$$\cos\lambda = \frac{p}{\sqrt{p^2+q^2+r^2}}, \qquad \cos\mu = \frac{q}{\sqrt{p^2+q^2+r^2}}, \qquad \cos\nu = \frac{r}{\sqrt{p^2+q^2+r^2}},$$

which are the angles that the diameter through the fixed point makes with the axes of coordinates.

V. *Tangent planes drawn to a series of concyclic ellipsoids through the points where they are met by a common diameter, envelope an hyperbolic cylinder.*

Let

$$a^2 \xi^2 + b^2 v^2 + c^2 \zeta^2 = 1, \tag{a}$$

be the equation of one of the ellipsoids;

$$x = mz, \qquad y = nz, \tag{b}$$

the projective equations of the fixed diameter, then

$$x = a^2\xi, \qquad y = b^2 v, \qquad z = c^2\zeta, \tag{c}$$

see (25), and as the surfaces are concyclic,

$$\frac{1}{b^2} = \frac{1}{a^2} + \frac{1}{h^2}, \qquad \frac{1}{c^2} = \frac{1}{a^2} + \frac{1}{h'^2}. \tag{d}$$

Now from the eight equations (a), (b), (c), (d), eliminating the six quantities $x, y, z, a, b, c$, we shall have two resulting equations between $\xi, v, \zeta, m, n, h, h'$, one of which will be found to be the tangential equation of an infinitely distant point, and the other is that of an hyperboloid, which two simultaneous equations denote an hyperbolic cylinder.

VI. *Three right lines constituting a right angled trihedral angle, revolve round a fixed point in space, meeting a surface of the second order in three points; the plane passing through those points envelopes a surface of revolution* (E) *of the second order, whose focus is the given point, and whose directrix plane relative to this focus is the polar plane of the fixed point relative to the given surface* ($\Sigma$).

Let the fixed point be taken as origin, then the projective equation of ($\Sigma$) being

$$\text{A}x^2 + \text{A}'y^2 + \text{A}''z^2 + 2\text{B}yz + 2\text{B}'xz + 2\text{B}''xy + 2\text{C}x + 2\text{C}'y + 2\text{C}''z = 1. \tag{a}$$

Let $r =$ length of one of the revolving lines, $\lambda, \mu, \nu$ the angles it makes with the axes, then $x = r\cos\lambda$, $y = r\cos\mu$, $z = r\cos\nu$, let the equation of the plane passing through $(x, y, z)$ be

$$x\xi + yv + z\zeta = 1. \tag{b}$$

Substituting in (a) and (b) the values of $x, y, z$, we find

$$(\text{A}\cos^2\lambda + \text{A}'\cos^2\mu + \text{A}''\cos^2\nu + 2\text{B}\cos\mu\cos\nu + 2\text{B}'\cos\lambda\cos\nu + 2\text{B}''\cos\lambda\cos\mu)$$
$$+ 2(\text{C}\cos\lambda + \text{C}'\cos\mu + \text{C}''\cos\nu)\frac{1}{r} = \frac{1}{r^2},$$

and

$$(\xi\cos\lambda + v\cos\mu + \zeta\cos\nu) = \frac{1}{r};$$

eliminating $r$ between those equations, we obtain

$$\left.\begin{aligned}
&(\text{A}\cos^2\lambda + \text{A}'\cos^2\mu + \text{A}''\cos^2\nu + 2\text{B}\cos\mu\cos\nu + 2\text{B}'\cos\lambda\cos\nu + 2\text{B}''\cos\lambda\cos\mu)\\
&+ 2(\text{C}\xi\cos^2\lambda + \text{C}'v\cos^2\mu + \text{C}''\zeta\cos^2\nu) + 2\text{C}(v\cos\lambda\cos\mu + \zeta\cos\lambda\cos\nu)\\
&+ 2\text{C}'(\zeta\cos\mu\cos\nu + \xi\cos\mu\cos\lambda) + 2\text{C}''(\xi\cos\nu\cos\lambda + v\cos\nu\cos\mu)\\
&= \xi^2\cos^2\lambda + v^2\cos^2\mu + \zeta^2\cos^2\nu + 2\xi v\cos\lambda\cos\mu + 2v\zeta\cos\mu\cos\nu\\
&\qquad + 2\xi\zeta\cos\nu\cos\lambda;
\end{aligned}\right\} \tag{c}$$

finding precisely similar expressions for the two other lines, adding the three equations together, and introducing the six relations of the nine angles, we get for the tangential equation of the enveloped surface

$$\xi^2 + v^2 + \zeta^2 - 2\mathrm{c}\,\xi - 2\mathrm{c}'\,v - 2\mathrm{c}''\,\zeta = \mathrm{A} + \mathrm{A}' + \mathrm{A}''; \qquad \text{(d)}$$

put $\mathrm{A} + \mathrm{A}' + \mathrm{A}'' = \frac{1}{a}$ and the equation is transformed into

$$a\,\xi^2 + a\,v^2 + a\,\zeta^2 - 2a\,\mathrm{c}\,\xi - 2a\,\mathrm{c}'\,v - 2a\,\mathrm{c}'' = 1. \qquad \text{(e)}$$

Now as the coefficients of the squares of the variables are equal, and the rectangles vanish; (e) is the equation of a surface of revolution whose focus is the origin. (See page 14).

The cosines of the angles which the major axis of this surface makes with the axes of coordinates are by (33)

$$\frac{\mathrm{c}}{\sqrt{\mathrm{c}^2+\mathrm{c}'^2+\mathrm{c}''^2}}, \qquad \frac{\mathrm{c}'}{\sqrt{\mathrm{c}^2+\mathrm{c}'^2+\mathrm{c}''^2}}, \qquad \frac{\mathrm{c}''}{\sqrt{\mathrm{c}^2+\mathrm{c}'^2+\mathrm{c}''^2}};$$

but these are the cosines of the angles which a perpendicular $\mathrm{P}$ from the fixed point, on the polar plane of this point relative to ($\Sigma$), makes with the axes of coordinates, hence the major axis of (E) coincides with this perpendicular.

This plane is the directrix plane of (E), for if $a$ and $b$ are the semi-axes of (E), $b^2 = a$, $a^2 = a + a^2(\mathrm{c}^2 + \mathrm{c}'^2 + \mathrm{c}''^2)$;

$$\therefore \frac{a^2 - b^2}{b^4} = \mathrm{c}^2 + \mathrm{c}'^2 + \mathrm{c}''^2 = \frac{1}{\mathrm{P}^2}; \text{ or } \mathrm{P} = \frac{a}{e} - ae.$$

Should the fixed point be assumed on the surface of ($\Sigma$), the absolute term of equation (a) then vanishes, and (e) is changed into

$$2\mathrm{c}\,a\,\xi + 2\mathrm{c}'\,a\,v + 2\mathrm{c}''\,a\,\zeta + 1 = 0, \qquad \text{(f)}$$

the tangential equation of a point. Suppose the tangent plane at the given point on the surface of ($\Sigma$) assumed as the plane of $x\,y$; then $\mathrm{c} = 0$, $\mathrm{c}' = 0$, in (a); and (f) is changed into $2\mathrm{c}''\zeta + \mathrm{A} + \mathrm{A}' + \mathrm{A}'' = 0$, hence the point is on the normal at the distance $\frac{-2\mathrm{c}''}{\mathrm{A} + \mathrm{A}' + \mathrm{A}''}$, from the surface, which particular case of the general theorem has been long known.

VII. *A series of surfaces of the second order touch seven given planes, the locus of their centres is a plane.*

Let the tangential equation of one of the surfaces be

$$a\xi^2 + a'v^2 + a''\zeta^2 + 2\beta v\zeta + 2\beta'\xi\zeta + 2\beta''\xi v + 2\gamma\xi + 2\gamma' v + 2\gamma''\zeta = 1.$$

Now there are seven linear equations to determine the nine coefficients,

we may then eliminate any six, and connect the three remaining by a linear equation, eliminating $\alpha, \alpha', \alpha'', \beta, \beta', \beta''$, we find

$$\text{L}\gamma + \text{M}\gamma' + \text{N}\gamma'' = 1;$$

the projective equation of a plane, where L, M, N, are determinate functions of the twenty-one constant tangential coordinates of the seven given planes; but it has been shown (29), that $\gamma, \gamma', \gamma''$, are the projective coordinates of the centre.

When there are eight given planes, seven of the coefficients may be eliminated, hence

$$\gamma = \text{H}\gamma'' + \text{K}, \qquad \gamma' = \text{H}'\gamma'' + \text{K}',$$

the projective equations of a right line.

## IV.

### ON THE TANGENTIAL EQUATIONS OF THE PARABOLOIDS.

Let the axis of the surface be taken as the axis of $x$, the tangents to the vertices of the principal sections, as those of $y$ and $z$, then it may be shown, that the tangential equation of the paraboloids is

$$l'\zeta^2 + l\upsilon^2 + \xi = 0; \tag{37}$$

$l$ and $l'$ being the one-fourth of the parameters of the principal sections in the planes of $yx$ and $zx$.

Let the surface now be referred to any three rectangular axes passing through the vertex, by the aid of formulæ (21); and (37) manifestly assumes the form

$$f\xi^2 + f'\upsilon^2 + f''\zeta^2 + g\zeta\upsilon + g'\xi\zeta + g''\xi\upsilon + h\xi + h'\upsilon + h''\zeta = 0, \tag{38}$$

let $\lambda, \mu, \nu$, be the angles which the axis of the surface makes with the axes of coordinates, then

$$\cos\lambda = hw, \quad \cos\mu = h'w, \quad \cos\nu = h''w; \quad w = (h^2 + h'^2 + h''^2)^{-\frac{1}{2}}, \tag{39}$$

$l'$ and $l$ are given by the quadratic equation

$$\rho^2 - (f + f' + f'')\rho + [(ff' + f'f'' + f''f) - (g^2 + g'^2 + g''^2)] = 0, \tag{40}$$

should the surface be referred to any three rectangular axes in space, the equation of the surface retains the form (38), the inclination of the axis is still given by (39), but the expressions for the parameters, and the coordinates of the vertex are too complicated to find a place here.

EXAMPLE.

*The reciprocal polar of any surface of the second order, the centre of the directrix surface being on the given surface, is a paraboloid.*

The directrix surface being for simplicity a sphere, whose radius is unity, at whose centre the origin of coordinates is placed, let the projective equation of the given surface be

$$\text{A}x^2+\text{A}'y^2+\text{A}''z^2+2\text{B}yz+2\text{B}'xz+2\text{B}''xy+2\text{C}x+2\text{C}'y+2\text{C}''z=0,$$

the tangential equation of its reciprocal polar is (see page 17)

$$\text{A}\xi^2+\text{A}'v^2+\text{A}''\zeta^2+2\text{B}v\zeta+2\text{B}'\xi\zeta+2\text{B}''\xi v+2\text{C}\xi+2\text{C}'v+2\text{C}''\zeta=0,$$

the tangential equation of a paraboloid.

## V.

### ON THE CAUSTIC BY REFLEXION OF A CIRCLE.*

Let $\rho$ and $c$ denote the reciprocals of the radius of the given circle, and of the distance of the radiant point from its centre, at which the origin of coordinates is placed, the axis of $x$ passing through the radiant point, then combining the principle of the equality of the angles of incidence and reflexion with the equations $\xi x + v y = 1$, and

$$\rho^2 (x^2 + y^2) = 1,$$

we shall find, after some very complicated eliminations, for the tangential equation of the caustic,

$$[(2\rho)^2 - (\xi + c)^2] (\xi^2 + v^2) = 4\rho^2 (\rho^2 - c\xi). \qquad \text{(a)}$$

Solving this equation for $v$, we find

$$v = \frac{2\rho^2 - \xi(\xi + c)}{\sqrt{(2\rho)^2 - (\xi + c)^2}}; \qquad \text{(b)}$$

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* The general solution of this problem long baffled the skill of the most expert analysts; at length M. Gergonne announced, *Annales de Mathématiques*, tom. xv. page 346, "Jétais, depuis quelque temps, en possession de l'équation de la caustique par réflexion sur le cercle, qui n'avait encore été donnée par personne; mais je l'avais obtenu par des calculs trop prolixes, et sous une forme trop peu elegante pour songer à la publier," &c.

Some time after the complete solution was given in the seventeenth volume of the same work by M. de St. Laurent, but in a most complicated and unmanageable form.

hence giving to $\xi$ a system of values from $0$ to $(2\rho - c)$, we find the corresponding values of $v$, so that the curve may be described by the successive intersections of right lines drawn according to a given law.

The tangent to the caustic drawn from the radiant point, is also a tangent to the circle, for when $\xi = c$, $v^2 = \rho^2 - c^2$.

Let a point be assumed on the axis of $x$, at the negative side of the origin, at the distance $\frac{1}{c}$ from the centre, the lines drawn from this point to the extremities of the diameter of the circle which coincides with the axis of $y$, are tangents to the caustic, for if $\xi = -c$, $v = \pm\rho$.

The tangents to the caustic parallel to the axis of $x$, intersect the axis of $y$ at the distance $\frac{1}{\rho}\sqrt{1 - \frac{c^2}{4\rho^2}}$ from the centre, for when

$$\xi = 0, \quad v = \frac{\rho}{\sqrt{1 - \frac{c^2}{4\rho^2}}}, \tag{c}$$

to find the distances of the tangents to the caustic parallel to the axis of $y$ from the centre, in this case $v = 0$, and we shall have to determine the corresponding values of $\xi$, the biquadratic equation

$$\xi^4 + 2c\,\xi^3 + (c^2 - 4\rho^2)\,\xi^2 - 4\rho^2 c\,\xi + 4\rho^4 = 0, \tag{d}$$

which is the square of the expression

$$\xi^2 + c\,\xi - 2\rho^2 = 0;$$

solving this equation, we find that the reciprocals of the distances of the tangents parallel to the axis of $y$, from the centre, are equal to

$$\frac{-c \pm \sqrt{c^2 + 8\rho^2}}{2}. \tag{e}$$

We find where the ray indefinitely near the axis of $x$ meets that axis by putting $v = \infty$, hence from (b),

$$\xi + c = \pm 2\rho,$$

which is the common formula for finding the principal focus.

The two particular cases where $c = 0$, and $c = \rho$, may be easily deduced from the above.

When the reflecting surface is a sphere, the tangential equation of the caustic surface is

$$[(2\rho)^2 - (\xi + c)^2]\,(\xi^2 + v^2 + \zeta^2) = 4\rho^2\,(\rho^2 - c\xi).$$

## VI.

### ON THE SURFACE OF THE CENTRES OF CURVATURE* OF AN ELLIPSOID.

Let the tangential equation of an ellipsoid be (see page 12)

$$a^2\,\xi^2 + b^2\,v^2 + c^2\,\zeta^2 = 1,$$

then the tangential equation of the surface of curvature is

$$\left(\xi^2+v^2+\zeta^2\right)^2 = \left(\frac{\xi^2}{a^2}+\frac{v^2}{b^2}+\frac{\zeta^2}{c^2}\right)\left(a^2\xi^2+b^2v^2+c^2\zeta^2-1\right). \quad \text{(a)}$$

From this equation it manifestly follows, that if parallel tangent planes are drawn to the surface of curvature, and to the ellipsoid, the difference of the squares of the perpendiculars from the centre on those planes, is always equal to the square of the coincident semi-diameter of the ellipsoid.

Reducing equation (a), it assumes the form

$$\left.\begin{array}{l}(a^2-b^2)^2c^2\xi^2v^2+(b^2-c^2)^2a^2\zeta^2v^2+(a^2-c^2)^2b^2\xi^2\zeta^2 \\ \quad -(b^2c^2\xi^2+a^2c^2v^2+a^2b^2\zeta^2)=0.\end{array}\right\} \quad \text{(b)}$$

*The surface of curvature has a cusp on the umbilical normal of the ellipsoid.*

For if the tangential coordinates of the umbilical normal, $\xi=\phi$, $\zeta=\phi'$; $\phi, \phi'$, being functions of the axes of the ellipsoid, are substituted in equation (b), the resulting value of $v$ assumes the form $\frac{0}{0}$, hence an infinite number of tangent planes may be drawn through the umbilical normal to the surface of curvature; this surface has then four cusps situated on the umbilical normals of the ellipsoid, consequently the surface whose projective equation is

$$\begin{array}{l}(a^2-b^2)^2\,c^2\,x^2\,y^2+(b^2-c^2)^2\,a^2\,y^2\,z^2+(a^2-c^2)^2\,b^2\,x^2\,z^2 \\ \quad -r^4\,(b^2\,c^2\,x^2+a^2\,c^2\,y^2+a^2\,b^2\,z^2)=0,\end{array}$$

has four curves of plane contact, with so many tangent planes. (See page 18.)

The sections of the surface made by the coordinate planes are found

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* The consideration of this surface, first imagined by Monge, but not discussed by him, will be found to throw some light on the nature of the *umbilici*, and of the lines of curvature passing through them, relative to which there has been some diversity of opinion. On this subject see Monge, *Application de l'Analyse a la Géométrie;* Dupin, *Développements de Géométrie*, pp. 173—187; Poisson, *Journal de l'Ecole Polytechnique*, 21ᵉ cahier, p. 205.

by making each variable successively infinite and zero; thus the sections of the surface made by the plane of $xy$ are found by putting $\zeta = \infty$, and $\zeta = 0$, let

$$b^2 - c^2 = b^2\eta^2, \qquad a^2 - c^2 = a^2\varepsilon^2, \qquad a^2 - b^2 = a^2 e^2;$$

the sections in the plane of $xy$ are

$$a^2\varepsilon^4\xi^2 + b^2\eta^4 v^2 = 1, \tag{c}$$

and

$$b^2\xi^2 + a^2 v^2 = a^4 e^4 \xi^2 v^2, \tag{d}$$

of which equations, the former is that of an ellipse, the latter that of the evolute of the section of the ellipsoid made by the plane of $xy$.

When the ellipsoid is one of revolution $b = c$, or $\eta = 0$, and the ellipse (c) degenerates into a right line coincident with the axis of the surface as is otherwise evident.

There are several other properties of this surface, which want of space compels us to omit.

Resuming equation (d), let $\xi\,\text{X} = 1$, $v\,\text{Y} = 1$, $ae^2 = \text{A}$, $\dfrac{a^2e^2}{b} = \text{B}$. Substituting those values in (d), the tangential equation of the evolute of a conic section whose semi-axes are $a$, $b$, assumes the remarkable form

$$\frac{\text{X}^2}{\text{A}^2} + \frac{\text{Y}^2}{\text{B}^2} = 1; \tag{e}$$

from the form of this equation several properties of the evolute may be deduced.

Let a right line revolve between two rectangular axes, so that the square of the segment of one axe, plus $n$ times the square of the segment of the other, may be constant; the right line envelopes the evolute of a conic section.

The line which joins the feet of the ordinates let fall from any point of a conic section on its axes, envelopes the evolute of a conic section, for if $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ be the equation of the section, $x = \text{X}$, $y = \text{Y}$, and this equation is changed into $\dfrac{\text{X}^2}{a^2} + \dfrac{\text{Y}^2}{b^2} = 1$, the tangential equation of the evolute of a conic section.

Let two points $m$, $n$, be assumed on the base $\alpha\beta$, of the evolute, $o$ being the centre, so that $\alpha n \times n\beta = \overline{om}^2$, and if from $m$ and $n$ two tangents are drawn to the evolute of which the segments intercepted by the other axe $o\gamma$, are $s$ and $s'$,

$$s^2 + s'^2 = A^2 + B^2,$$

for

$$s^2 = \frac{B^2}{A^2}(A^2 - X^2) + X^2, \qquad s'^2 = \frac{B^2}{A^2}(A^2 - X'^2) + X'^2;$$

but by supposition $A^2 = X^2 + X'^2$, hence $s^2 + s'^2 = A^2 + B^2$.

Let two tangents, mutually at right angles, be drawn to the evolutes of two confocal conic sections, one to each $\varsigma, \varsigma'$, denoting the reciprocals of the segments of the tangents made by the axes, then

$$\varsigma^2 + \varsigma'^2 = \frac{1}{A^2} + \frac{1}{B'^2}.$$

For let

$$a^2 - b^2 = k^2, \quad \text{then} \quad \frac{1}{A^2} - \frac{1}{B^2} = \frac{1}{k^2}, \qquad \frac{1}{A'^2} - \frac{1}{B'^2} = \frac{1}{k^2};$$

hence

$$\varsigma^2 = \frac{1}{B^2} + \frac{\cos^2\omega}{k^2}, \qquad \varsigma'^2 = \frac{1}{B'^2} + \frac{\sin^2\omega}{k^2},$$

or

$$\varsigma^2 + \varsigma'^2 = \frac{1}{A^2} + \frac{1}{B'^2}.$$

*The normals of a conic section whose equation is* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, *are augmented by a constant quantity k, adjacent to the curve, the tangential equation of the curve thus generated is*

$$[(a^2 - k^2)\,\xi^2 + (b^2 - k^2)\,v^2 - 1]^2 = 4k^2(\xi^2 + v^2);$$

hence generally, the extremity of a cord (whose length when unwound off a quadrant of the evolute of a conic section is $l$,) will generate a curve whose tangential equation is of the fourth order,

There is one particular case where the curve so generated will be a conic section; when $l = \frac{B^3}{B^2 - A^2}$, A, B, being the semi-axes of the evolute.

## VII.

### ON THE RECTIFICATION OF PLANE CURVES.

When the tangential equation of a curve is given, a formula for rectification may be immediately deduced, for let P be the perpendicular from the origin on the tangent, $\lambda$ the angle it makes with the axis of $x$, then $\xi = \frac{\cos\lambda}{P}$, $v = \frac{\sin\lambda}{P}$, and the equation of the curve $\Phi(\xi, v) = 0$ is transformed into $P = \phi(\lambda)$; substituting this value of P in the formula

$$s = u + \int \mathrm{P}\, d\lambda,$$

we obtain

$$s = u_1 - u_0 + \int_{\lambda_0}^{\lambda_1} d\lambda \,.\, \phi(\lambda); \tag{a}$$

$u$, the orthogonal projection of the radius vector on the tangent not being supposed to change sign between the limits of integration.

For example :—*The tangential equation of the reciprocal polar of the elliptic lemniscate is*

$$(\xi^2 + v^2)^2 = \frac{\xi^2}{a^2} + \frac{v^2}{b^2}, \quad \text{or} \quad \frac{1}{\mathrm{P}^2} = \frac{\cos^2\lambda}{a^2} + \frac{\sin^2\lambda}{b^2},$$

or

$$\mathrm{P} = \frac{b}{\sqrt{1 - e^2 \sin^2\lambda}};$$

hence

$$s = u_1 - u_0 + b \int_{\lambda_0}^{\lambda_1} d\lambda \frac{1}{\sqrt{1 - e^2 \sin^2\lambda}},$$

the arc of which curve is therefore a general geometrical representation of elliptic functions of the first order.

*The tangential equation of the involute of the hypocycloid whose projective equation is*

$$\left(\frac{x}{r}\right)^{\frac{2}{3}} + \left(\frac{y}{r}\right)^{\frac{2}{3}} = 1,$$

will be found to be

$$\left[l\,\xi^2 + \left(l - \frac{r}{2}\right) v^2\right]^2 = \xi^2 + v^2,$$

$l$ being the distance of the generating point from the centre in its initial position on the axis of $x$.

This curve is rectifiable, for if $\mathrm{P}$ be the perpendicular from the centre on the tangent; $\mathrm{P} = l - \frac{r}{2} \sin^2\lambda$.

When $r$ may be neglected compared with $l$, the equation is that of a circle.

*The tangential equation of the cycloid.*

Let the origin be placed at the centre of the base, then its tantial equation is

$$\Delta = v + \xi \tan^{-1}\left(\frac{\xi}{v}\right),$$

where $\Delta$ is the reciprocal of the diameter of the generating circle,

$$P\,\Delta = (\cos\lambda + \lambda\,\sin\lambda),$$

$\lambda$ being measured from the vertical axis; hence the cycloid is rectifiable.

*Epicycloids and hypocycloids are rectifiable;* for let $r$ be the radius of the rolling circle, $2nr$ that of the base circle, the initial position of the generating point being on the axis of $x$.

$$\xi = \frac{\sin(n \pm 1)\,\omega}{2r(n \pm 1)\sin n\omega}, \qquad v = \frac{\cos(n \pm 1)\,\omega}{2r(n \pm 1)\sin n\omega};$$

when $\omega$ may be eliminated between those equations, we obtain the tangential equation of the curve,

now $$\frac{1}{P^2} = \xi^2 + v^2, \quad \text{or} \quad P = 2r(n \pm 1)\sin n\omega,$$

and $$(n \pm 1)\,\omega + \lambda = \frac{\pi}{2}; \text{ hence } \int P\,d\lambda = \frac{2r(n \pm 1)^2}{n}\cos n\omega.$$

The equation $F(x,y)=0$, is the integral of the equation $\Phi(\xi,v)=0$; for if between the three equations $\Phi(\xi,v)=0$, $x\xi + yv = 1$, and $\frac{\xi}{v} = -\frac{dy}{dx}$ we eliminate $\xi$, and $v$, $\Phi(\xi,v)=0$, is changed into $\Phi\left(x, y, \frac{dy}{dx}\right) = 0$, of which $F(x,y)=0$, is the integral; hence, if by any means we can arrive at the projective and tangential equations of the same curve, we shall have obtained the integral of a certain differential equation of the first order and $n^{\text{th}}$ degree.

The confined limits of this essay preclude the possibility of discussing by this method the theory of the singular points of curves and curved surfaces, cusps, multiple points, and points of inflexion, to which it may with much facility be applied; or of investigating the properties of other curves, algebraic and transcendental, which flow from the consideration of their tangential equations.

Owing to the same cause, I have been compelled to rest satisfied in many cases with barely hinting at principles, which being novel, would require perhaps further elucidation; and to suppress several of the demonstrations, which however by an attentive reader may be easily supplied.

THE END.